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AN EQUATION FOR SOUND VELOCITY EXPRESSED EXPLICITLY AS A FUNCTION OF PRESSURE

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The velocity of sound in real gases is commonly expressed in terms of the density and the virial coefficients. However, an equation for the sound velocity given explicitly in terms of the pressure (a quantity more readily measurable than the density) is not available. The essence of this paper is to develop such an equation.

1. INTRODUCTION

The expression of the sound velocity V for real gases in common use is the one derived as a function of the density ρ and the virial coefficients B, C and D and their derivatives [1, 2]

$$V^2 = \frac{RT}{M}\left[\Phi + \frac{\Psi}{X}(\gamma^0 - 1)\right], \tag{1}$$

where

$$\Phi = 1 + 2B\rho + 3C\rho^2 + 4D\rho^3 + \ldots,$$

$$\Psi = \left[1 + \left(B + T\frac{\mathrm{d}B}{\mathrm{d}T}\right)\rho + \left(C + T\frac{\mathrm{d}C}{\mathrm{d}T}\right)\rho^2 + \left(D + T\frac{\mathrm{d}D}{\mathrm{d}T}\right)\rho^3\right]^2,$$

$$X = 1 - (\gamma^0 - 1)\left[\left(2T\frac{\mathrm{d}B}{\mathrm{d}T} + T^2\frac{\mathrm{d}^2B}{\mathrm{d}T^2}\right)\rho + \frac{1}{2}\left(2T\frac{\mathrm{d}C}{\mathrm{d}T} + T^2\frac{\mathrm{d}^2C}{\mathrm{d}T^2}\right)\rho^2 + \frac{1}{3}\left(2T\frac{\mathrm{d}D}{\mathrm{d}T} + T^2\frac{\mathrm{d}^2D}{\mathrm{d}T^2}\right)\rho^3\right].$$

R is the gas constant; T the temperature; M the molecular weight; γ the ratio of heat capacities; the superscript 0 denotes the condition of the ideal gas state; B, C, D are the virial coefficients appearing in the virial equation of state

$$Z = P/R\rho T = 1 + B\rho + C\rho^2 + D\rho^3 + \cdots. \tag{2}$$

Here P is the pressure; and Z the compressibility factor.

When the virial coefficients and their derivatives are available, equation (1) can be used to calculate the velocity of sound at a given T and ρ. However, if the velocity of sound is to be calculated at a given T and P, a trial solution for ρ is first required before equation (1) can be used. To overcome this difficulty an equation for the velocity of sound expressed explicitly as a function of T and P must be found.

Let the equation of state be expressed as a power series in pressure

$$Z = P/R\rho T = 1 + B'P + C'P^2 + D'P^3 + \cdots, \tag{3a}†$$

where [3]

$$B' = B/RT, \qquad C' = (C - B^2)/(RT)^2, \qquad D' = (D - 3CB - 2B^3)/(RT)^3. \tag{3b}$$

† The coefficients B', C', D', like the virial coefficients B, C, D are functions of temperature alone, and can be analytically shown to be related to the virial coefficients according to equation (3b).

2. THE NEW DERIVED EXPRESSION

The velocity of sound is given by:

$$V^2 = \gamma \frac{\partial P}{\partial \rho}\Big)_T = -\gamma v^2 \frac{\partial P}{\partial v}\Big)_T, \tag{4}$$

where v denotes the specific volume.

It is rather difficult to carry out the desired derivation starting with equation (4) as it stands, inasmuch as (ρ, T) rather than (p, T) are the independent variables, but when the reciprocal of equation (4) is taken to be the starting point, the derivation becomes easy and straightforward with (p, T) as independent variables. Hence equation (4) is rewritten as

$$\frac{1}{V^2} = -\frac{1}{\gamma v^2} \frac{\partial v}{\partial P}\Big)_T . \tag{5}$$

First, it is desired to obtain an expression for $1/\gamma$. This can be done with the aid of equation (6) below [3],

$$C_p - C_v = -T\left[\frac{\partial v}{\partial T}\Big)_p\right]^2 \frac{\partial P}{\partial v}\Big)_T, \tag{6}$$

in which C_p and C_v are the heat capacities at constant pressure and constant volume, respectively.

From equation (6)

$$\frac{C_v}{C_p} = \frac{1}{\gamma} = 1 + \frac{T}{C_p}\left[\frac{\partial v}{\partial T}\Big)_p\right]^2 \frac{\partial P}{\partial v}\Big)_T . \tag{7}$$

When equation (7) is substituted into equation (5), therefore

$$\frac{1}{V^2} = -\frac{1}{v^2}\left\{\frac{\partial v}{\partial P}\Big)_T + \frac{T}{C_p}\left[\frac{\partial v}{\partial T}\Big)_p\right]^2\right\}. \tag{8}$$

Now, it becomes a matter of expressing the quantities $\partial v/\partial P)_T$, $\partial v/\partial T)_p$ and C_p in terms of pressure and temperature.

From the relation

$$\frac{\partial C_p}{\partial P} = -T \frac{\partial^2 v}{\partial T^2}\Big)_p$$

and equation (3a), C_p can be found to be

$$C_p = C_p^0 U,$$

where

$$U = 1 - \frac{\gamma^0 - 1}{\gamma^0}\left[\left(2T\frac{\mathrm{d}B'}{\mathrm{d}T} + T^2 \frac{\mathrm{d}^2 B'}{\mathrm{d}T^2}\right)p + \frac{1}{2}\left(2T\frac{\mathrm{d}C'}{\mathrm{d}T} + T^2\frac{\mathrm{d}^2 C'}{\mathrm{d}T^2}\right)p^2 + \right.$$
$$\left. + \frac{1}{3}\left(2T\frac{\mathrm{d}D'}{\mathrm{d}T} + T^2\frac{\mathrm{d}^2 D'}{\mathrm{d}T^2}\right)p^3 + \cdots\right].$$

Also with the aid of equation (3), the quantities $\partial v/\partial P)_T$ and $\partial v/\partial T)_p$ can be evaluated. Hence equation (5) becomes

$$\frac{1}{V^2} = \frac{M}{Z^2 RT}\left[S - \frac{(\gamma^0 - 1)\, Y}{\gamma^0\, U}\right] \tag{9}$$

and

$$S = 1 - C'p^2 - 2D'p^3 - \cdots$$

$$Y = \left[1 + \left(B' + T\frac{\mathrm{d}B'}{\mathrm{d}T}\right)p + \left(C' + T\frac{\mathrm{d}C'}{\mathrm{d}T}\right)p^2 + \left(D' + T\frac{\mathrm{d}D'}{\mathrm{d}T}\right)p^3 + \cdots\right]^2.$$